

On the pedant tree-connectivity of graphs *

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Abstract

The concept of pedant tree-connectivity was introduced by Hager [10] in 1985. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an S -Steiner tree or a Steiner tree connecting S (or simply, an S -tree) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. For an S -Steiner tree, if the degree of each vertex in S is equal to one, then this tree is called a *pedant S -Steiner tree*. Two pedant S -Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local pedant-tree connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pedant S -Steiner trees in G . For an integer k with $2 \leq k \leq n$, k -pedant tree-connectivity is defined as $\tau_k(G) = \min\{\tau_G(S) \mid S \subseteq V(G), |S| = k\}$. In this paper, we first study the sharp bounds of pedant tree-connectivity. Next, we obtain the exact value of a threshold graph, and give an upper bound of the pedant-tree k -connectivity of a complete multipartite graph. For a connected graph G , we show that $0 \leq \tau_k(G) \leq n - k$, and graphs with $\tau_k(G) = n - k, n - k - 1, n - k - 2, 0$ are characterized in this paper. In the end, we obtain the Nordhaus-Guadalupe type results for pedant tree-connectivity.

Keywords: connectivity, Steiner tree, pedant S -Steiner tree, packing, pedant tree-connectivity.

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1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [3] for graph theoretical notation and terminology not described here. For a graph G , let $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices, the set of edges and the minimum

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degree of G , respectively. Connectivity is one of the most basic concepts of graph-theoretic subjects, both in combinatorial sense and the algorithmic sense. As we know, the classical connectivity has two equivalent definitions. The *connectivity* of G , written $\kappa(G)$, is the minimum order of a vertex set $S \subseteq V(G)$ such that $G \setminus S$ is disconnected or has only one vertex. We call this definition the ‘cut’ version definition of connectivity. A well-known theorem of Whitney provides an equivalent definition of connectivity, which can be called the ‘path’ version definition of connectivity. For any two distinct vertices x and y in G , the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y . Then $\kappa(G) = \min\{\kappa_G(x, y) \mid x, y \in V(G), x \neq y\}$ is defined to be the *connectivity* of G . For connectivity, Oellermann gave a survey paper on this subject; see [28].

Although there are many elegant and powerful results on connectivity in Graph Theory, the basic notation of classical connectivity may not be general enough to capture some computational settings. So people want to generalize this concept. For the ‘cut’ version definition of connectivity, we find the above minimum vertex set without regard the number of components of $G \setminus S$. Two graphs with the same connectivity may have differing degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n}$ and the path P_{n+1} ($n \geq 3$) are both trees of order $n+1$ and therefore connectivity 1, but the deletion of a cut-vertex from $K_{1,n}$ produces a graph with n components while the deletion of a cut-vertex from P_{n+1} produces only two components. Chartrand et al. [4] generalized the ‘cut’ version definition of connectivity. For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the *k-connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G of order n ($n \geq k$) produces a graph with at least k components or a graph with fewer than k vertices. Thus, for $k = 2$, $\kappa'_2(G) = \kappa(G)$. For more details about k -connectivity, we refer to [4, 7, 28, 29].

The generalized connectivity of a graph G , introduced by Hager [10], is a natural generalization of the ‘path’ version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an *S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Note that when $|S| = 2$ a Steiner tree connecting S is just a path connecting the two vertices of S . Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint trees connecting S in G , that is, we search for the maximum cardinality of edge-disjoint trees which include S and are vertex disjoint with the exception of S . For an integer k with $2 \leq k \leq n$, *generalized k-connectivity* (or *k-tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all k -subsets of $V(G)$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G , that is, $\kappa_2(G) = \kappa(G)$, which

is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G . By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. Note that the generalized k -connectivity and k -connectivity of a graph are indeed different. Take for example, the graph H_1 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. Then $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$. There are many results on the generalized connectivity, see [5, 15, 17, 18, 19, 20, 21, 22, 23, 24, 30].

The concept of pedant-tree connectivity [10] was introduced by Hager in 1985, which is specialization of generalized connectivity (or k -tree-connectivity) but a generalization of classical connectivity. For an S -Steiner tree, if the degree of each vertex in S is equal to one, then this tree is called a *pedant S -Steiner tree*. Two pedant S -Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local-pedant tree connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pedant S -Steiner trees in G . For an integer k with $2 \leq k \leq n$, *pedant-tree k -connectivity* is defined as $\tau_k(G) = \min\{\tau_G(S) \mid S \subseteq V(G), |S| = k\}$. Set $\kappa_k(G) = 0$ when G is disconnected.

In [10], Hager derived the following results.

Lemma 1 [10] *Let G be a graph. If $\tau_k(G) \geq \ell$, then $\delta(G) \geq k + \ell - 1$.*

Lemma 2 [10] *Let G be a graph. If $\tau_k(G) \geq \ell$, then $\kappa(G) \geq k + \ell - 2$.*

Li et al. [19] obtained the following result.

Lemma 3 [19] *Let G be a connected graph with minimum degree δ . If there are two adjacent vertices of degree δ , then $\kappa(G) \leq \delta(G) - 1$.*

Obviously, the generalized k -connectivity (or k -tree-connectivity) and pedant-tree k -connectivity of a graph are indeed different. For example, let $H = W_n$ be a wheel of order n . From Lemma 1, we have $\tau_3(H) \leq 1$. One can check that for any $S \subseteq V(H)$ with $|S| = 3$, $\tau_H(S) \geq 1$. Therefore, $\tau_3(H) = 1$. From Lemma 3, we have $\kappa_3(H) \leq \delta(H) - 1 = 3 - 1 = 2$. One can check that for any $S \subseteq V(G)$ with $|S| = 3$, $\kappa_H(S) \geq 2$. Therefore, $\kappa_3(H) = 2$.

In addition to being a natural combinatorial measure, both the pendant tree-connectivity and the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G , then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and

popularly used in the physical design of VLSI (see [8, 9, 32]) and computer communication networks (see [6]). Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k -connectivity can serve for measuring the capability of a network G to connect any k vertices in G .

This paper is organized as follows. In Section 2, we study sharp bounds of pedant tree-connectivity. In Section 3, we obtain the exact value of a threshold graph, and give an upper bound of the pedant-tree k -connectivity of a complete multipartite graph. For a connected graph G , we show that $0 \leq \tau_k(G) \leq n - k$, and graphs with $\tau_k(G) = n - k, n - k - 1, n - k - 2$ are characterized in Section 4. By Fan Lemma, graphs with $\tau_k(G) = 0$ are characterized in Section 5.

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n and $\mathcal{G}(n, m)$ the subclass of $\mathcal{G}(n)$ having m edges. Give a graph theoretic parameter $f(G)$ and a positive integer n , the *Nordhaus-Gaddum(N-G) Problem* is to determine sharp bounds for: (1) $f(G) + f(\bar{G})$ and (2) $f(G) \cdot f(\bar{G})$, as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide investigations. Recently, Aouchiche and Hansen published a survey paper on this subject, see [1]. In Section 6, we study the Nordhaus-Gaddum problem for pedant tree-connectivity.

2 Sharp bounds of pedant tree-connectivity

In [10], Hager obtained the exact value of the pedant-tree k -connectivity of a complete graph.

Lemma 4 [10] *Let k, n be two integers with $3 \leq k \leq n$, and let K_n be a complete graph of order n . Then*

$$\tau_k(K_n) = n - k.$$

Let S be a set of k vertices of a connected graph G , and let \mathcal{T} be a set of internally disjoint pedant S -Steiner trees. Then the following observation is immediate.

Observation 1 *Let k, n be two integers with $3 \leq k \leq n$. Let G be a connected graph of order n , and let $S \subseteq V(G)$ with $|S| = k$. For each $T \in \mathcal{T}$,*

$$|E(T) \cap E_G[S, \bar{S}]| \geq k,$$

where $\bar{S} = V(G) \setminus S$.

From the above result, we can derive an upper bound of k -pedant tree-connectivity.

Theorem 1 For any graph G with order at least k ,

$$\tau_k(G) \leq \min_{S \subseteq V(G), |S|=k} \left\lfloor \frac{1}{k} |E_G[S, \bar{S}]| \right\rfloor,$$

where $S \subseteq V(G)$ with $|S| = k$, and $\bar{S} = V(G) \setminus S$. Moreover, the bound is sharp.

Proof. For any $S \subseteq V(G)$ with $|S| = k$, it suffices to $\tau(S) = |\mathcal{T}| \leq \frac{1}{k} |E_G[S, \bar{S}]|$. From Observation 1, for each tree $T \in \mathcal{T}$, we have

$$|E(T) \cap E_G[S, \bar{S}]| \geq k.$$

Therefore,

$$k|\mathcal{T}| \leq |E_G[S, \bar{S}]|.$$

Since $|\mathcal{T}|$ is an integer, we have

$$\tau(S) = |\mathcal{T}| \leq \left\lfloor \frac{1}{k} |E_G[S, \bar{S}]| \right\rfloor.$$

From the arbitrariness of S , we have

$$\tau_k(G) = \min_{S \subseteq V(G), |S|=k} \tau(S) = \min_{S \subseteq V(G), |S|=k} |\mathcal{T}| \leq \min_{S \subseteq V(G), |S|=k} \left\lfloor \frac{1}{k} |E_G[S, \bar{S}]| \right\rfloor.$$

To show the bound is sharp, we consider the graph $G = K_n$. From Lemma 4, we have $\tau_k(G) = n - k$. Thus $|E_{K_n}[S, \bar{S}]| = k(n - k)$ and hence $\frac{1}{k} |E_{K_n}[S, \bar{S}]| = n - k$. Therefore,

$$\min_{S \subseteq V(G), |S|=k} \left\lfloor \frac{1}{k} |E_G[S, \bar{S}]| \right\rfloor = n - k = \tau_k(G).$$

So the bound of this theorem is sharp. For $k = 3$, the graph $H = W_n$ is a sharp example. As we know, $\tau_3(W_n) = 1$. Observe that the graph $H = W_n$ is obtained from a cycle $C = v_1 v_2 \cdots v_{n-1} v_1$ by adding a vertex v_n and the edges $v_i v_n$ ($1 \leq i \leq n - 1$). Choose $S = \{v_2, v_3, v_4\}$. Then $E_H[S, \bar{S}] = \{v_1 v_2, v_2 v_n, v_3 v_n, v_4 v_n, v_4 v_5\}$ and hence we have $\left\lfloor \frac{1}{3} |E_H[S, \bar{S}]| \right\rfloor = \left\lfloor \frac{1}{3} \cdot 5 \right\rfloor = 1 = \tau_3(H)$. \blacksquare

For any connected graph G of order n , from Lemma 4, we have $\tau_k(G) \leq \tau_k(K_n) = n - k$. The following upper and lower bounds for $\tau_k(G)$ can be easily seen.

Proposition 1 Let k, n be two integers with $3 \leq k \leq n$, and let G be a graph. Then

$$0 \leq \tau_k(G) \leq n - k.$$

For $k = n$, the following corollary is immediate.

Corollary 1 Let G be a graph of order n . Then $\tau_n(G) = 0$ if and only if G is a graph of order n .

3 Pedant-tree connectivity of some special graphs

In [10], Hager obtained the exact value of pedant tree-connectivity of a complete bipartite graph.

Lemma 5 [10] *Let $K_{r,s}$ be a complete bipartite graph with $r + s$ vertices. Then*

$$\tau_k(K_{r,s}) = \max\{\min\{r - k + 1, s - k + 1\}, 0\}.$$

For complete multipartite graphs, we obtain the following result.

Theorem 2 *Let K_{n_1, n_2, \dots, n_t} be a complete t -partite graph with $n_1 \leq n_2 \leq \dots \leq n_t$.*

(1) *If $k \geq t$, then*

$$\tau_k(K_{n_1, n_2, \dots, n_t}) \leq \left\lfloor \frac{\sum_{i=1}^t n_i - k}{2} \right\rfloor.$$

(2) *If $k < t$, then*

$$\tau_k(K_{n_1, n_2, \dots, n_t}) \leq \sum_{i=k+1}^t n_i + \left\lfloor \frac{\sum_{i=1}^k n_i - k}{2} \right\rfloor.$$

Moreover, the upper bounds are sharp.

Proof. Set $G = K_{n_1, n_2, \dots, n_t}$. Let V_1, V_2, \dots, V_t be the parts of complete t -partite graph G , and set $|V_i| = n_i$ ($1 \leq i \leq t$).

(1) Suppose $k \geq t$. Choose $S \subseteq V(G)$ and $|S| = k$ such that $S \cap V_i \neq \emptyset$ ($1 \leq i \leq t$). Observe that any pedant S -Steiner tree must muse at least two vertices in $\bar{S} = V(G) \setminus S$. Therefore,

$$\tau_k(G) \leq \tau(S) \leq \left\lfloor \frac{\sum_{i=1}^t n_i - k}{2} \right\rfloor.$$

(2) Suppose $k < t$. Choose $S \subseteq V(G)$ and $|S| = k$ such that $S \cap V_i \neq \emptyset$ ($1 \leq i \leq k$). Then $S \cap V_i = \emptyset$ ($k + 1 \leq i \leq t$). Observe that any pedant S -Steiner tree must muse at least one vertex in $\bar{S} = V(G) \setminus S$. Since $n_1 \leq n_2 \leq \dots \leq n_t$, it follows that there are at most $\sum_{i=k+1}^t n_i$ pedant S -Steiner trees in G . Any other pedant S -Steiner tree must use at least two vertices in \bar{S} . Therefore,

$$\tau_k(G) \leq \tau(S) \leq \sum_{i=k+1}^t n_i + \left\lfloor \frac{\sum_{i=1}^k n_i - k}{2} \right\rfloor.$$

The proof is now complete. ■

To show the sharpness of the bound in Theorem 2, we consider the following example.

Example 1: Let $G = K_{r,r,r}$ be complete 3-part graph where $r = n_1 = n_2 = n_3$. Suppose $k = 3$. From the above theorem, $\tau_3(G) \leq \lfloor \frac{3r-3}{2} \rfloor$. To show $\tau_3(G) \geq \lfloor \frac{3r-3}{2} \rfloor$, it suffices to prove that $\tau(S) \geq \lfloor \frac{3r-3}{2} \rfloor$ for any $S \subseteq V(G)$ and $|S| = 3$. Let $U = \{u_1, u_2, \dots, u_r\}$, $V = \{v_1, v_2, \dots, v_r\}$ and $W = \{w_1, w_2, \dots, w_r\}$ be the three parts of G . Suppose $|S \cap U| = 3$ or $|S \cap V| = 3$ or $|S \cap W| = 3$. Without loss of generality, let $|S \cap U| = 3$. Then the trees T_i induced by the edges in $\{v_i u_1, v_i u_2, v_i u_3\}$ ($1 \leq i \leq r$) and the trees T'_j induced by the edges in $\{w_j u_1, w_j u_2, w_j u_3\}$ ($1 \leq j \leq r$) are $2r > \lfloor \frac{3r-3}{2} \rfloor$ internally disjoint pedant S -Steiner trees, which implies $\tau(S) \geq \lfloor \frac{3r-3}{2} \rfloor$, as desired. Suppose $|S \cap U| = 2$ or $|S \cap V| = 2$ or $|S \cap W| = 2$. Without loss of generality, let $|S \cap U| = 2$. Then $|S \cap V| = 1$ or $|S \cap W| = 1$. Without loss of generality, let $S = \{u_1, u_2, v_1\}$. Then the trees T_i induced by the edges in $\{w_i u_1, w_i u_2, w_i u_3\}$ ($1 \leq i \leq r$) and the trees T'_j induced by the edges in $\{v_j u_1, v_j u_2, v_j v_1, u_j v_j\}$ ($3 \leq j \leq r$) are $r + \lfloor \frac{r-2}{2} \rfloor \geq \lfloor \frac{3r-3}{2} \rfloor$ internally disjoint pedant S -Steiner trees, and hence $\tau(S) \geq \lfloor \frac{3r-3}{2} \rfloor$, as desired. Suppose $|S \cap U| = |S \cap V| = |S \cap W| = 1$. Without loss of generality, let $S = \{u_r, v_r, w_r\}$. Then the trees T_i induced by the edges in $\{v_i u_r, u_i v_r, u_i v_i, w_i w_r, v_i w_r\}$ and the trees T'_i induced by the edges in $\{v_{i+1} u_r, v_{i+1} w_r, v_{i+1} w_i, w_i v_r\}$ and the trees T''_i induced by the edges in $\{u_{i+1} w_r, u_{i+1} w_{i+1}, w_{i+1} v_r, w_{i+1} u_r\}$ are $\lfloor \frac{3r-3}{2} \rfloor$ internally disjoint pedant S -Steiner trees, and hence $\tau(S) \geq \lfloor \frac{3r-3}{2} \rfloor$, where $i = 2\ell - 1$ and $1 \leq \ell \leq \lfloor \frac{r-1}{2} \rfloor$. From the above argument, $\tau_3(G) = \lfloor \frac{3r-3}{2} \rfloor$, which implies that the bound of (1) of Theorem 2 is sharp. For the bound of (2) of Theorem 2, one can check that the complete 4-partite graph $K_{r,r,r,r}$ is a sharp example.

A graph G is a *threshold graph*, if there exists a weight function $w : V(G) \rightarrow R$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$.

The following observation is easy to make from the definition of a threshold graph.

Observation 2 Let $G([n], E)$ be a threshold graph with a weight function $w : V(G) \rightarrow R$. Let the vertices be labelled so that $w(1) \geq w(2) \geq \dots \geq w(n)$. Then

- (a) $d_1 \geq d_2 \geq \dots \geq d_n$, where d_i is the degree of vertex i .
- (b) $I = \{i \in V(G) : d_i \leq i - 1\}$ is a maximum independent set of G and $G \setminus I$ is a clique in G .
- (c) $N(i) = \{1, 2, \dots, d_i\}$ for every $i \in I$. Thus, the neighborhoods of vertices in I form a linear order under set inclusion. Furthermore, if G is connected, then every vertex in G is adjacent to 1.

Now, we are in a position to give our result.

Theorem 3 *Let G be a threshold graph with $\delta(G) = \ell$. Then*

$$\tau_k(G) = \begin{cases} 0, & \text{if } k > \ell; \\ \ell - k + 1, & \text{if } k \leq \ell. \end{cases}$$

Proof. Let C_r and I_{n-r} denote the clique and the maximum independent set of G , respectively. Set $V(C_r) = \{u_1, u_2, \dots, u_r\}$ and $V(I_{n-r}) = \{v_1, v_2, \dots, v_{n-r}\}$. Since $\delta(G) = \ell$, it follows that $v_j u_i \in E(G)$ for each u_i ($1 \leq i \leq \ell$) and each v_j ($1 \leq j \leq n-r$).

Suppose $k \geq \ell + 1$. Choose $S = \{u_1, u_2, \dots, u_{k-1}, v_{n-r}\}$. Observe that $v_{n-r} u_i \in E(G)$ ($1 \leq i \leq \ell$) and $v_{n-r} u_i \notin E(G)$ ($\ell + 1 \leq i \leq r$). Then any pedant S -Steiner tree must occupy some edge $v_{n-r} u_j \in E(G)$ ($1 \leq j \leq \ell$). Since $u_j \in S$ ($1 \leq j \leq \ell$), it follows that the degree of u_j in this tree is at least two, a contradiction. So $\tau_k(G) \leq 0$. Combining this with Proposition 1, we have $\tau_k(G) = 0$.

Suppose $k \leq \ell$. Choose $S = \{u_1, u_2, \dots, u_{k-1}, v_{n-r}\}$. Note that $v_{n-r} u_i \in E(G)$ ($1 \leq i \leq \ell$) and $v_{n-r} u_i \notin E(G)$ ($\ell + 1 \leq i \leq r$). Then any pedant S -Steiner tree T must occupy some edge $v_{n-r} u_j \in E(G)$ ($1 \leq j \leq \ell$). If $v_{n-r} u_j \in E(T)$ ($1 \leq j \leq k-1$), then the degree of u_j in T is at least two, a contradiction. We now assume $v_{n-r} u_j \in E(G)$ ($k \leq j \leq \ell$). Because a pedant S -Steiner tree must occupy at least one edge in $\{v_{n-r} u_j \mid k \leq j \leq \ell\}$, we have at most $\ell - k + 1$ pedant S -Steiner trees in G , which implies $\tau_k(G) \leq \ell - k + 1$.

From the definition of $\tau_k(G)$, it suffices to show that $\tau(S) \geq \ell - k + 1$ for any $S \subseteq V(G)$ and $|S| = k$. Suppose $S \subseteq V(C_r)$. Set $S = \{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$. From Lemma 4, $\tau_k(C_r) = r - k \geq \ell - k$, and hence there are $\ell - k$ pedant S -Steiner trees in G . These trees together with the tree induced by the edges in $\{u_{i_1} v_1, u_{i_2} v_1, \dots, u_{i_k} v_1\}$ are $\ell - k + 1$ pedant S -Steiner trees in G , as desired. Suppose $S \subseteq V(I_{n-r})$. Since $u_j v_{i_t} \in E(G)$ ($1 \leq j \leq \ell, 1 \leq t \leq k$), it follows that the trees induced by the edges in $\{v_{i_1} u_j, v_{i_2} u_j, \dots, v_{i_k} u_j\}$ are ℓ pedant S -Steiner trees in G , as desired. Suppose $S \cap V(C_r) \neq \emptyset$ and $S \cap V(I_{n-r}) \neq \emptyset$. Set $|S \cap V(C_r)| = p$ and $|S \cap V(I_{n-r})| = k - p$. Clearly, $1 \leq p \leq k - 1$ and $|S \cap \{u_1, u_2, \dots, u_\ell\}| \leq p$. Then there exist at least $\ell - p$ vertices belonging to $V(G) \setminus S$. Choose $\ell - k + 1$ vertices from them, say $u_{j_1}, u_{j_2}, \dots, u_{j_{\ell-k+1}}$. Set $S \cap V(C_r) = \{u_{i_1}, u_{i_2}, \dots, u_{i_p}\}$. Without loss of generality, let $S \cap V(I_{n-r}) = \{v_1, v_2, \dots, v_{k-p}\}$. Then the trees induced by the edges in $\{u_{i_1} u_{j_i}, u_{i_2} u_{j_i}, \dots, u_{i_p} u_{j_i}\} \cup \{v_1 u_{j_i}, v_2 u_{j_i}, \dots, v_{k-p} u_{j_i}\}$ ($1 \leq i \leq \ell - k + 1$) are $\ell - k + 1$ pedant S -Steiner trees in G , as desired. From the above argument, we know that $\tau_k(G) = \ell - k + 1$. \blacksquare

4 Graphs with large pedant tree-connectivity

The graphs attaining the upper bound of Proposition 1 can be characterized now.

Theorem 4 *Let k, n be two integers with $3 \leq k \leq n$ and $n \geq 4$, and let G be a connected graph. Then $\tau_k(G) = n - k$ if and only if G is a complete graph of order n .*

Proof. Suppose $\tau_k(G) = n - k$. From Lemma 1, we have $\delta(G) \geq k + (n - k) - 1 = n - 1$. So G is a complete graph of order n . Conversely, suppose G is a complete graph of order n . For any $S \subseteq V(G)$ with $|S| = k$, we have $|V(G) \setminus S| = n - k$. Let $S = \{u_1, u_2, \dots, u_k\}$ and $V(G) \setminus S = \{w_1, w_2, \dots, w_{n-k}\}$. Then the trees T_i induced by the edges in $\{w_i u_1, w_i u_2, \dots, w_i u_k\}$ ($1 \leq i \leq n - k$) are $n - k$ internally disjoint pedant S -Steiner tree, which implies that $\tau(S) \geq n - k$. From the arbitrariness of S , we have $\tau_k(G) \geq n - k$. Combining this with Proposition 1, $\tau_k(G) = n - k$. ■

The graphs with $\tau_k(G) = n - k - 1$ can also be characterized in the following.

Theorem 5 *Let k, n be two integers with $3 \leq k \leq n$ and $n \geq 7$, and let G be a connected graph. Then $\tau_k(G) = n - k - 1$ if and only if $\bar{G} = rK_2 \cup (n - 2r)K_1$ ($r = 1, 2$).*

Proof. Suppose $\tau_k(G) = n - k - 1$. From Lemma 1, we have $\delta(G) \geq k + (n - k - 1) - 1 = n - 2$. Therefore, G is a graph obtained from a complete graph of order n by deleting a matching M such that $0 \leq |M| \leq \lfloor \frac{n}{2} \rfloor$. Then we have the following claim.

Claim 1. $1 \leq |M| \leq 2$.

Proof of Claim 1. Set $|M| = r$. Then $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Assume $r \geq 3$. Without loss of generality, let $M = \{u_i w_i \mid 1 \leq i \leq r\}$. Choose $S = \{u_1, u_2, u_3, v_1, v_2, \dots, v_{k-3}\} \subseteq V(G)$ where $v_1, v_2, \dots, v_{k-3} \in V(G) \setminus \{u_1, u_2, u_3, w_1, w_2, w_3\}$. Thus $u_i w_i \notin E_G[S, \bar{S}]$ ($1 \leq i \leq 3$), where $\bar{S} = V(G) \setminus S$. From the definition of $\tau_k(G)$, any pedant S -Steiner tree must use at least one vertex of \bar{S} . Observe that any pedant S -Steiner tree containing vertex w_i ($1 \leq i \leq 3$) must occupy at least two vertices of $\{w_1, w_2, w_3\}$ since $u_i w_i \notin E_G[S, \bar{S}]$ ($1 \leq i \leq 3$). So w_1, w_2, w_3 must belong to two pedant S -Steiner tree, say T_1, T_2 , which implies that these trees occupy at least four vertices of $\bar{S} = V(G) \setminus S$. So there are at most $n - k - 2$ internally disjoint pedant S -Steiner trees, a contradiction.

From Claim 1, we have $1 \leq |M| \leq 2$, as desired.

Conversely, suppose that G is a connected graph of order n satisfying the conditions of this theorem. It suffices to show $\tau_k(G) \geq n - k - 1$, where $G = K_n \setminus M$ and M is a matching of size 1 or 2. In fact, we only need to prove $\tau_k(G) \geq n - k - 1$, where $G = K_n \setminus M$ and M is a matching of size 2. From the definition of $\tau_k(G)$, it suffices to show that $\tau(S) \geq n - k - 1$ for any $S \subseteq V(G)$ with $|S| = k$. Since $|M| = 2$, it follows that $0 \leq |M \cap E_{K_n}[S, \bar{S}]| \leq 2$.

If $|M \cap E_{K_n}[S, \bar{S}]| = 2$, then we set $M = \{u_1 v_1, u_2 v_2\}$ and $V(G) \setminus \{u_1, u_2, v_1, v_2\} = \{w_1, w_2, \dots, w_{n-4}\}$. Without loss of generality, let $S = \{u_1, u_2, w_1, w_2, \dots, w_{k-2}\}$. Then

$v_1, v_2 \in \bar{S}$. Then the trees T_j induced by the edges in $\{w_j u_1, w_j u_2, w_j w_1, v_j w_2, \dots, w_j u_{k-2}\}$ ($k-1 \leq j \leq n-4$) together with the tree T_1 induced by the edges in $\{v_1 u_2, v_1 w_1, v_1 w_2, \dots, v_1 w_{k-2}, v_1 v_2, v_2 u_1\}$ form $(n-4) - (k-2) + 1 = n-k-1$ pedant S -Steiner trees, which implies $\tau(S) \geq n-k-1$.

If $|M \cap E_{K_n}[S, \bar{S}]| = 1$, then we set $u_1 v_1 \in M \cap E_{K_n}[S, \bar{S}]$ and $V(G) \setminus \{u_1, v_1\} = \{w_1, w_2, \dots, w_{n-2}\}$. Without loss of generality, let $S = \{u_1, w_1, w_2, \dots, w_{k-1}\}$. Then the trees T_j induced by the edges in $\{w_j u_1, w_j w_1, w_j w_2, \dots, w_j u_k\}$ ($k \leq j \leq n-2$) form $(n-4) - (k-1) + 2 = n-k-1$ pedant S -Steiner trees, and hence $\tau(S) \geq n-k-1$.

If $|M \cap E_{K_n}[S, \bar{S}]| = 0$, then we let $V(G) = \{w_1, w_2, \dots, w_n\}$. Without loss of generality, let $S = \{w_1, w_2, \dots, w_k\}$. Then the trees T_j induced by the edges in $\{w_j w_1, w_j w_2, \dots, w_j w_k\}$ ($k+1 \leq j \leq n-4$) together with the trees T_i induced by the edges in $\{w_1 u_i, w_2 u_i, \dots, w_k u_i\}$ ($1 \leq i \leq 2$) and the trees T'_j induced by the edges in $\{w_1 v_j, w_2 v_j, \dots, w_k v_j\}$ ($1 \leq j \leq 2$) form $(n-4) - k + 2 + 2 = n-k$ pedant S -Steiner trees, which implies that $\tau(S) \geq n-k$.

From the above argument, we conclude that for any $S \subseteq V(G)$ with $|S| = k$, $\tau(S) \geq n-k-1$. From the arbitrariness of S , we have $\tau_k(G) \geq n-k-1$. Combining this with Theorem 4, $\tau_k(G) = n-k-1$. \blacksquare

For $k = 3$, graphs with $\tau_k(G) = n-k-2$ are characterized in the following lemma, which is preparation of Theorem 6.

Lemma 6 *Let G be a connected graph of order n .*

(1) *For $k = 3$, $\tau_k(G) = n-k-2$ if and only if \bar{G} is a subgraph of one of the following graphs.*

- $C_i \cup C_j \cup (n-i-j)K_1$ ($i = 3, 4, j = 3, 4$);
- $C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$);
- $P_5 \cup \lfloor \frac{n-5}{2} \rfloor K_2$;
- $C_i \cup (n-i)K_1$ ($i = 5, 6, 7$).

(2) *For $k = 4$, $\tau_k(G) = n-k-2$ if and only if \bar{G} is a subgraph of one of the following graphs.*

- $C_i \cup C_j \cup (n-i-j)K_1$ ($i = 3, 4, j = 3, 4$);
- $C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$);
- $C_5 \cup K_2 \cup (n-7)K_1$;

- $C_i \cup (n-i)K_1$ ($i = 6, 7$).

Proof. Suppose $\tau_k(G) = n - k - 2$ ($k = 3, 4$). We first give the proof of the following claim.

Claim 1. For any $S \subseteq V(G)$ and $|S| = k$, if $|N_{\bar{G}}(S)| \geq 5$, then $\tau_k(G) \leq n - k - 3$.

Proof of Claim 1: Assume, to the contrary, that $\tau_k(G) \geq n - k - 2$. Choose five vertices in $N_{\bar{G}}(S)$, say w_1, w_2, w_3, w_4, w_5 . Observe that any pedant S -Steiner tree containing each vertex w_i ($1 \leq i \leq 5$) must occupy at least two vertices of \bar{S} . Another fact is that from the definition of $\tau_k(G)$, any pedant S -Steiner tree must use at least one vertex of \bar{S} . So the total number of the internally disjoint pedant S -Steiner trees is at most $n - k - 3$, as desired. ■

Let $H_1 = P_3 \cup P_3 \cup K_2$ and $H_2 = P_5 \cup P_3$. From Claim 1, for $k = 3$, \bar{G} contains neither H_1 and nor H_2 as its subgraph. Let $H_3 = P_3 \cup P_3 \cup K_2 \cup K_1$ and $H_4 = P_5 \cup P_3 \cup K_1$. From Claim 1, for $k = 4$, \bar{G} contains neither H_3 and nor H_4 as its subgraph. Furthermore, we have the facts as follows.

- \bar{G} contains at most two cycles.
- \bar{G} contains at most two paths of order at least 3.
- if \bar{G} contains a cycle, then the order of this cycle is at most 7.
- if \bar{G} contains a path of order at least 3, then the order of this path is at most 7.

We distinguish the following cases to show this lemma. Firstly, we suppose that exactly two components of \bar{G} are a union of two cycles, or two paths of order at least 3, or one is a cycle and the other is a path of order at least 3. Consider the case that \bar{G} contains exactly two cycles. Since \bar{G} does not contain H_2 as its subgraph, it follows that the order of each cycle in \bar{G} is at most 4, and each of other components is a isolated vertex except these cycles. Therefore,

$$\bar{G} = C_i \cup C_j \cup (n - i - j)K_1,$$

where $i = 3, 4$ and $j = 3, 4$. Consider the case that \bar{G} contains exactly two paths of order at least 3. Since \bar{G} does not contain H_2 as its subgraph, it follows that the order of each path in \bar{G} is at most 4, and each of other components is a isolated vertex except these paths. Therefore,

$$\bar{G} = P_i \cup P_j \cup (n - i - j)K_1,$$

where $i = 3, 4$ and $j = 3, 4$. Consider the case that \bar{G} contains a cycle and a path of order at least 3. Since \bar{G} does not contain H_2 as its subgraph, it follows that the order of this cycle in \bar{G} is at most 4 and the order of this path in \bar{G} is at most 4. Observe that each of

other components is a isolated vertex except these paths. Therefore,

$$\bar{G} = C_i \cup P_j \cup (n - i - j)K_1,$$

where $i = 3, 4$ and $j = 3, 4$. From the above argument, we know that \bar{G} is a subgraph of $C_i \cup C_j \cup (n - i - j)K_1$ ($i = 3, 4, j = 3, 4$), as desired.

Next, we suppose that \bar{G} contains exactly one cycle or one path of order at least 3. Since \bar{G} does not contain H_1 as its subgraph, if \bar{G} contains exactly one cycle, then the order of the unique cycle is at most 7. Furthermore, if the order of the unique cycle is 6 or 7, then each of other components is a isolated vertex except this path. Therefore, $\bar{G} = C_i \cup (n - i)K_1$, where $i = 6, 7$. Similarly, if the order of the unique path is 6 or 7, then each of other components is a isolated vertex except this cycle. Therefore, $\bar{G} = P_i \cup (n - i)K_1$, where $i = 6, 7$. Hence, \bar{G} is a subgraph of $C_i \cup (n - i)K_1$ ($i = 6, 7$).

For $k = 3$, we suppose the order of the unique cycle C is 5. Let $C = v_1v_2v_3v_4v_5v_1$. If there is an independent edge w_1w_2 in \bar{G} , then we choose $S = \{v_1, v_3, w_1\}$. Observe that any pedant S -Steiner tree occupy at least one vertex in $\bar{S} = V(G) \setminus S$. Thus there are at most $n - 7$ pedant S -Steiner trees by the vertices in $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5, w_1, w_2\}$. If there is no pedant S -Steiner tree containing w_2 , then any pedant S -Steiner tree must occupy three vertices in $V(C) \setminus S$, and hence there are at most $n - 6$ pedant S -Steiner trees in G , a contradiction. Suppose that there exists a pedant S -Steiner tree containing w_2 , say T . Then the tree T must occupy at most one vertex in $V(C) \setminus S$. Observe that there is no other pedant S -Steiner trees, and hence there are at most $n - 6$ pedant S -Steiner trees in G , also a contradiction. So each of other components is a isolated vertex except this cycle. Therefore, \bar{G} is a subgraph of $C_5 \cup (n - 5)K_1$. From the above argument, we know that \bar{G} is a subgraph of $C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$) or $C_j \cup (n - j)K_1$ ($j = 5, 6, 7$) or $P_5 \cup \lfloor \frac{n-5}{2} \rfloor K_2$.

For $k = 4$, we suppose the order of the unique path P is 5. Let $P = v_1v_2v_3v_4v_5$. If there are two independent edges w_1w_2 and w_3w_4 in \bar{G} , then we choose $S = \{w_1, w_3, v_2, v_4\}$. Then $|N_{\bar{G}}(S)| \geq 5$. From Claim 1, $\tau_4(G) \leq n - 7$, a contradiction. So there exists at most one nontrivial component in \bar{G} except this cycle. Therefore, \bar{G} is a subgraph of $C_5 \cup K_2 \cup (n - 7)K_1$. From the above argument, we know that \bar{G} is a subgraph of $C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$) or $C_j \cup (n - j)K_1$ ($j = 6, 7$) or $C_5 \cup K_2 \cup (n - 7)K_1$.

In the end, we suppose that \bar{G} contains no cycle and no path of order at least 3. Therefore, \bar{G} is a subgraph of $\lfloor \frac{n}{2} \rfloor K_2$, and hence \bar{G} is a subgraph of $C_4 \cup \lfloor \frac{n-4}{2} \rfloor K_2$, as desired.

Conversely, suppose that \bar{G} is a subgraph of the graphs in this lemma. For $k = 3$, it suffices to show that $\tau_3(G) = n - 5$. In fact, we only need to show that $\tau_3(G) = n - 5$ for

- $\bar{G} = C_i \cup C_j \cup (n - i - j)K_1$ ($i = 3, 4, j = 3, 4$);

- $\bar{G} = C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$);
- $\bar{G} = P_5 \cup \lfloor \frac{n-5}{2} \rfloor K_2$;
- $\bar{G} = C_i \cup (n-i)K_1$ ($i = 5, 6, 7$).

Suppose $\bar{G} = C_4 \cup C_4 \cup (n-8)K_1$. Let $C = u_1u_2u_3u_4u_1$ and $C' = v_1v_2v_3v_4v_1$ be the cycles in \bar{G} . It suffices to show that $\tau(S) \geq n-5$ for any $S \subseteq V(G)$ and $|S| = 3$. Set $S = \{x, y, z\}$, $R = \{u_i \mid 1 \leq i \leq 4\} \cup \{v_i \mid 1 \leq i \leq 4\}$ and $V(G) \setminus R = \{w_1, w_2, \dots, w_{n-8}\}$. If $|S \cap R| = 0$, then the trees T'_i ($1 \leq i \leq 4$) induced by the edges in $\{xu_i, yu_i, zu_i\}$, the trees T''_i ($1 \leq i \leq 4$) induced by the edges in $\{xv_i, yv_i, zv_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-3$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x, y, z\}$, as desired. Suppose $|S \cap R| = 1$. Without loss of generality, let $z = u_4$. Then the tree T'_2 induced by the edges in $\{xu_2, yu_2, zu_2\}$, the trees T''_i ($1 \leq i \leq 4$) induced by the edges in $\{xv_i, yv_i, zv_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x, y\}$, as desired. Suppose $|S \cap R| = 2$ and $|S \cap V(C)| = 2$. Without loss of generality, let $y = u_2$ and $z = u_4$. Then the trees T'_i ($1 \leq i \leq 4$) induced by the edges in $\{xv_i, yv_i, zv_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x\}$, as desired. Suppose $|S \cap R| = 2$ and $|S \cap V(C)| = |S \cap V(C')| = 1$. Without loss of generality, let $y = u_4$ and $z = v_4$. Then the trees T'_1 induced by the edges in $\{xu_1, yv_3, v_3u_1, u_1z\}$, the trees T'_2 ($1 \leq i \leq 4$) induced by the edges in $\{xv_1, yv_1, u_3v_1, u_3z\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x\}$, as desired. Suppose $|S \cap R| = 3$ and $|S \cap V(C)| = 3$. Then the trees T'_i ($1 \leq i \leq 4$) induced by the edges in $\{xv_i, yv_i, zv_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-4$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\}$, as desired. Suppose $|S \cap R| = 3$ and $|S \cap V(C)| = 2$. Without loss of generality, let $x = u_2$, $y = u_4$ and $z = v_4$. Then the tree T'_1 induced by the edges in $\{xv_3, yv_3, v_3u_3, u_3z\}$, the tree T'_2 induced by the edges in $\{xv_1, yv_1, u_1v_1, u_1z\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-4$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\}$, as desired.

Suppose $\bar{G} = C_7 \cup (n-7)K_1$. Let $C = u_1u_2 \dots u_7u_1$ be the cycle in \bar{G} . It suffices to show that $\tau(S) \geq n-5$ for any $S \subseteq V(G)$ and $|S| = 3$. Set $S = \{x, y, z\}$ and $V(G) \setminus V(C) = \{w_1, w_2, \dots, w_{n-7}\}$. If $|S \cap V(C)| = 0$, then the trees T'_i ($1 \leq i \leq 7$) induced by the edges in $\{xu_i, yu_i, zu_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-3$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\} \setminus \{x, y, z\}$, as desired. Suppose $|S \cap V(C)| = 1$. Without loss of generality, let $z = u_7$. Then the trees T'_i ($2 \leq i \leq 5$) induced by the edges in $\{xu_i, yu_i, zu_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\} \setminus \{x, y\}$, as desired. Suppose that $|S \cap V(C)| = 3$. We only need to consider the cases $\{x, y, z\} = \{u_1, u_2, u_3\}$, $\{x, y, z\} = \{u_1, u_2, u_4\}$,

$\{x, y, z\} = \{u_1, u_3, u_5\}$ and $\{x, y, z\} = \{u_1, u_2, u_5\}$. If $\{x, y, z\} = \{u_1, u_2, u_3\}$, then the trees T'_i ($i = 5, 6$) induced by the edges in $\{xu_i, yu_i, zu_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\}$, as desired. If $\{x, y, z\} = \{u_1, u_2, u_4\}$, then the tree T'_1 induced by the edges in $\{xu_6, yu_6, zu_6\}$, the tree T'_2 induced by the edges in $\{u_2u_7, u_4u_7, u_5u_7, u_1u_5\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\}$, as desired. If $\{x, y, z\} = \{u_1, u_3, u_5\}$, then the tree induced by the edges in $\{u_1u_4, u_4u_7, u_3u_7, u_7u_5\}$, the tree induced by the edges in $\{u_1u_6, u_2u_6, u_3u_6, u_2u_5\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\}$, as desired. If $\{x, y, z\} = \{u_1, u_2, u_5\}$, then the trees induced by the edges in $\{u_1u_6, u_2u_6, u_3u_6, u_3u_5\}$, the tree induced by the edges in $\{u_2u_4, u_1u_4, u_4u_7, u_7u_5\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\}$, as desired. Suppose $|S \cap V(C)| = 2$. We only need to consider the cases $\{x, y, z\} = \{w_1, u_3, u_4\}$, $\{x, y, z\} = \{w_1, u_2, u_4\}$ and $\{x, y, z\} = \{w_1, u_2, u_5\}$. If $\{x, y, z\} = \{w_1, u_2, u_3\}$, then the trees T'_i ($i = 1, 6, 7$) induced by the edges in $\{xu_i, yu_i, zu_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_2, w_3, \dots, w_{n-7}\}$, as desired. If $\{x, y, z\} = \{w_1, u_2, u_4\}$, then the trees T'_i ($i = 6, 7$) induced by the edges in $\{xu_i, yu_i, zu_i\}$, the tree T'' induced by the edges in $\{xu_1, u_1u_5, yu_4, zu_1\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_2, w_3, \dots, w_{n-7}\}$, as desired. If $\{x, y, z\} = \{w_1, u_2, u_5\}$, then the tree T'_1 induced by the edges in $\{xu_7, yu_7, zu_7\}$, the tree T'_2 induced by the edges in $\{w_1u_4, u_1u_4, u_2u_4, u_1u_5\}$, the tree T'_3 induced by the edges in $\{w_1u_3, u_3u_5, u_3u_6, u_2u_6\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees where $w_j \in \{w_2, w_3, \dots, w_{n-7}\}$, as desired.

Suppose $\bar{G} = P_5 \cup \lfloor \frac{n-5}{2} \rfloor K_2$. Let $P = u_1u_2u_3u_4u_5$ be the unique path in \bar{G} . It suffices to show that $\tau(S) \geq n-5$ for any $S \subseteq V(G)$ and $|S| = 3$. Set $S = \{x, y, z\}$ and $V(G) \setminus V(P) = \{w_1, w_2, \dots, w_{n-5}\}$. Note that $C_7 \cup \lfloor \frac{n-7}{2} \rfloor K_2$ is a subgraph of \bar{G} . Suppose that $|S \cap V(P)| = 3$ or $|S \cap V(P)| = 2$. Recall that we have checked that there are $n-5$ pedant S -Steiner trees in the complement of $C_7 \cup (n-7)K_1$. In fact, one can check that if $|S \cap V(P)| = 3$ or $|S \cap V(P)| = 2$, then there are $n-5$ pedant S -Steiner trees in the complement of $C_7 \cup \lfloor \frac{n-7}{2} \rfloor K_2$, and hence there are $n-5$ pedant S -Steiner trees in the complement of \bar{G} . We now assume that $|S \cap V(P_5)| = 1$ or $|S \cap V(P)| = 0$. If $|S \cap V(P)| = 0$, then the trees T'_i ($1 \leq i \leq 5$) induced by the edges in $\{xu_i, yu_i, zu_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-4$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-5}\} \setminus \{x, y, z, w_1\}$ where w_1 is a vertex adjacent to x in \bar{G} , as desired. Suppose $|S \cap V(P)| = 1$. Without loss of generality, let $z \in V(P)$ and $x, y \in V(G) \setminus V(P)$. If $xy \in E(\bar{G})$, then one can check that there are $n-5$ pedant S -Steiner trees in G since $C_7 \cup \lfloor \frac{n-7}{2} \rfloor K_2$ is a subgraph of \bar{G} . We may assume $xy \notin E(\bar{G})$. Without

loss of generality, let $x = w_1$, $y = w_3$, $w_1w_2 \in E(\bar{G})$ and $w_3w_4 \in E(\bar{G})$. If $z = u_1$, then the trees T'_i ($3 \leq i \leq 5$) induced by the edges in $\{xu_i, yu_i, zu_i\}$, the tree induced by the edges in $\{u_1w_2, w_2w_4, w_1w_4, w_2w_3\}$ and the trees T_j ($5 \leq i \leq n-5$) induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees, as desired. If $z = u_2$, then the trees T'_i ($4 \leq i \leq 5$) induced by the edges in $\{xu_i, yu_i, zu_i\}$, the tree induced by the edges in $\{u_2w_4, v_3w_4, u_3w_3, v_3w_1\}$, the tree induced by the edges in $\{u_2w_2, u_1w_2, w_2w_3, u_1w_1\}$ and the trees T_j ($5 \leq i \leq n-5$) induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees, as desired. If $z = u_3$, then the trees T'_i ($i = 1, 5$) induced by the edges in $\{xu_i, yu_i, zu_i\}$, the tree induced by the edges in $\{u_4w_3, w_1u_4, u_4w_4, w_4u_3\}$, the tree induced by the edges in $\{u_2w_1, w_2u_2, u_3w_2, w_2w_3\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j\}$ are $n-5$ pedant S -Steiner trees, as desired.

For other cases, one can also check that $\tau(S) \geq n-5$ for any $S \subseteq V(G)$ and $|S| = 3$. Therefore, $\tau_3(G) \geq n-5$. From Theorem 5, we have $\tau_3(G) = n-5$.

For $k = 4$, it suffices to show that $\tau_4(G) = n-6$. In fact, we only need to show that $\tau_4(G) = n-6$ for

- $\bar{G} = C_i \cup C_j \cup (n-i-j)K_1$ ($i = 3, 4$, $j = 3, 4$);
- $\bar{G} = C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$);
- $\bar{G} = P_5 \cup K_2 \cup (n-7)K_1$;
- $\bar{G} = C_i \cup (n-i)K_1$ ($i = 5, 6, 7$).

Suppose $\bar{G} = C_4 \cup C_4 \cup (n-8)K_1$. Let $C = u_1u_2u_3u_4u_1$ and $C' = v_1v_2v_3v_4v_1$ be the cycles in \bar{G} . It suffices to show that $\tau(S) \geq n-6$ for any $S \subseteq V(G)$ and $|S| = 4$. Set $S = \{x, y, z, r\}$, $R = \{u_i \mid 1 \leq i \leq 4\} \cup \{v_i \mid 1 \leq i \leq 4\}$ and $V(G) \setminus R = \{w_1, w_2, \dots, w_{n-8}\}$. If $|S \cap R| = 0$, then the trees T'_i ($1 \leq i \leq 4$) induced by the edges in $\{xu_i, yu_i, zu_i, ru_i\}$, the trees T''_i ($1 \leq i \leq 4$) induced by the edges in $\{xv_i, yv_i, zv_i, rv_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-4$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x, y, z, r\}$, as desired. Suppose $|S \cap R| = 1$. Without loss of generality, let $r = v_3$. Then the tree T' induced by the edges in $\{xv_1, yv_1, zv_1, rv_1\}$, the trees T''_i ($1 \leq i \leq 4$) induced by the edges in $\{xu_i, yu_i, zu_i, zu_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x, y, z\}$, as desired. Suppose $|S \cap R| = 2$ and $|S \cap V(C')| = 2$. Without loss of generality, let $r = v_1$ and $z = v_3$. Then the trees T'_i ($1 \leq i \leq 4$) induced by the edges in $\{xu_i, yu_i, zu_i, ru_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x, y\}$, as desired. Suppose $|S \cap R| = 2$ and $|S \cap V(C)| = |S \cap V(C')| = 1$. Without loss of generality, let $r = u_1$ and $z = v_3$. Then the tree T'_1 induced by the edges in $\{xu_2, yu_2, v_2u_2, u_2z, v_2r\}$, the tree T'_2 induced by the edges in

$\{xv_4, yv_4, rv_4, u_4v_4, u_4z\}$, the tree T'_3 induced by the edges in $\{xv_1, yv_1, zv_1, rv_1\}$, the tree T'_4 induced by the edges in $\{xu_3, yu_3, zu_3, ru_3\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x, y\}$, as desired. Suppose $|S \cap R| = 3$ and $|S \cap V(C)| = 3$. Without loss of generality, let $S \cap V(C) = \{y, z, r\}$. Then the trees T'_i ($1 \leq i \leq 4$) induced by the edges in $\{xv_i, yv_i, zv_i, rv_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x\}$, as desired. Suppose $|S \cap R| = 3$ and $|S \cap V(C)| = 2$. Without loss of generality, let $r = v_4$, $y = u_3$ and $z = u_4$. Then the trees T'_1 induced by the edges in $\{xv_3, yv_3, zv_3, v_3u_2, u_2r\}$, the trees T'_2 induced by the edges in $\{xu_1, yu_1, u_1r, u_1v_1, v_1z\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\} \setminus \{x\}$, as desired. Suppose $|S \cap R| = 4$ and $|S \cap V(C)| = 4$. Without loss of generality, let $S \cap V(C) = \{x, y, z, r\}$. Then the trees T'_i ($1 \leq i \leq 4$) induced by the edges in $\{xv_i, yv_i, zv_i, rv_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\}$, as desired. Suppose $|S \cap R| = 4$ and $|S \cap V(C)| = 3$. Without loss of generality, let $S \cap V(C) = \{x, y, z\}$. Then the tree T'_1 induced by the edges in $\{xv_2, yv_2, zv_2, rv_2\}$, the tree T'_2 induced by the edges in $\{xv_1, yv_1, zv_1, u_1v_1, ru_1\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\}$, as desired. Suppose $|S \cap R| = 4$ and $|S \cap V(C)| = 2$. Without loss of generality, let $x = u_2$, $y = u_4$, $z = v_2$ and $r = v_4$. Then the trees T'_1 induced by the edges in $\{xv_1, yv_1, v_1u_3, zu_3, u_3r\}$, the trees T'_2 induced by the edges in $\{xv_3, yv_3, u_1v_3, u_1r, u_1z\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-8}\}$, as desired.

Suppose $\bar{G} = C_7 \cup (n-7)K_1$. Let $C = u_1u_2 \dots u_7u_1$ be the cycle in \bar{G} . It suffices to show that $\tau(S) \geq n-6$ for any $S \subseteq V(G)$ and $|S| = 4$. Set $S = \{x, y, z, r\}$ and $V(G) \setminus V(C) = \{w_1, w_2, \dots, w_{n-7}\}$. If $|S \cap V(C)| = 0$, then the trees T'_i ($1 \leq i \leq 7$) induced by the edges in $\{xu_i, yu_i, zu_i, ru_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-4$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\} \setminus \{x, y, z, r\}$, as desired. Suppose $|S \cap V(C)| = 1$. Without loss of generality, let $z = u_1$. Then the trees T'_i ($3 \leq i \leq 6$) induced by the edges in $\{xu_i, yu_i, zu_i, ru_i\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_1, w_2, \dots, w_{n-7}\} \setminus \{x, y, z\}$, as desired. Suppose $|S \cap V(C)| = 2$. Without loss of generality, let $z = u_1$ and $r = u_4$. Then the tree T'_1 induced by the edges in $\{xu_6, yu_6, zu_6, ru_6\}$, the tree T'_2 induced by the edges in $\{xu_3, yu_3, zu_3, u_3u_7, ru_7\}$, the tree T'_3 induced by the edges in $\{xu_2, yu_2, ru_2, u_2u_5, zu_5\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n-6$ pedant S -Steiner trees where $w_j \in \{w_2, w_3, \dots, w_{n-7}\} \setminus \{x, y\}$, as desired. Suppose that $|S \cap V(C)| = 3$. Without loss of generality, let $y = u_1$, $z = u_4$ and $r = u_6$. Then the tree T' induced by the edges in $\{xu_5, yu_5, u_5u_2, zu_2, ru_2\}$, the tree T'' induced by the edges in $\{xu_3, yu_3, zu_7, u_3u_7, ru_3\}$ and

the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n - 6$ pedant S -Steiner trees where $w_j \in \{w_2, w_3, \dots, w_{n-7}\} \setminus \{x\}$, as desired. Suppose that $|S \cap V(C)| = 4$. Without loss of generality, let $x = u_1$, $y = u_3$, $z = u_5$ and $r = u_7$. Then the tree T' induced by the edges in $\{xu_6, yu_6, u_6u_2, zu_2, ru_2\}$ and the trees T_j induced by the edges in $\{xw_j, yw_j, zw_j, rw_j\}$ are $n - 6$ pedant S -Steiner trees where $w_j \in \{w_2, w_3, \dots, w_{n-7}\}$, as desired.

For other cases, one can also check that $\tau(S) \geq n - 6$ for any $S \subseteq V(G)$ and $|S| = 4$. Therefore, $\tau_4(G) \geq n - 6$. From Theorem 5, we have $\tau_4(G) = n - 6$. \blacksquare

For $3 \leq k \leq n$, graphs with $\tau_k(G) = n - k - 2$ can also be characterized in the following.

Theorem 6 *Let k, n be two integers with $3 \leq k \leq n$ and $n \geq 15$, and let G be a connected graph.*

(1) *For $5 \leq k \leq n$, $\tau_k(G) = n - k - 2$ if and only if $\bar{G} = P_3 \cup (n - 3)K_1$ or G satisfies all the following conditions.*

- $1 \leq \Delta(\bar{G}) \leq 2$;
- $e(\bar{G}) \geq 3$;
- *for any $R \subseteq V(G)$ with $|E_{\bar{G}}[w, \bar{R}]| \geq 1$ where $w \in R$ and $\bar{R} = V(G) \setminus R$, the size of R is at most 4.*

(2) *For $k = 3$, $\tau_k(G) = n - k - 2$ if and only if \bar{G} is a subgraph of one of the following graphs.*

- $C_i \cup C_j \cup (n - i - j)K_1$ ($i = 3, 4, j = 3, 4$);
- $C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$);
- $P_5 \cup \lfloor \frac{n-5}{2} \rfloor K_2$;
- $C_i \cup (n - i)K_1$ ($i = 5, 6, 7$).

(3) *For $k = 4$, $\tau_k(G) = n - k - 2$ if and only if \bar{G} is a subgraph of one of the following graphs.*

- $C_i \cup C_j \cup (n - i - j)K_1$ ($i = 3, 4, j = 3, 4$);
- $C_i \cup \lfloor \frac{n-i}{2} \rfloor K_2$ ($i = 3, 4$);
- $C_5 \cup K_2 \cup (n - 7)K_1$;
- $C_i \cup (n - i)K_1$ ($i = 6, 7$).

Proof. From Lemma 6, the conclusion is true for the case $k = 3, 4$. We now assume $5 \leq k \leq n$. Suppose $\tau_k(G) = n - k - 2$. If $e(\bar{G}) = 2$, then $\bar{G} = P_3 \cup (n - 3)K_1$ by Theorem 5. Conversely, we suppose $\bar{G} = P_3 \cup (n - 3)K_1$. One can check that there exist $n - k - 2$ pedant S -Steiner tree in G for any $S \subseteq V(G)$ and $|S| = k$. Then $\tau_k(G) \geq n - k - 2$. From Theorem 5, we have $\tau_k(G) = n - k - 2$.

From now on, we assume $e(\bar{G}) \geq 3$. Suppose $\tau_k(G) = n - k - 2$. From Lemma 1, we have $\delta(G) \geq k + (n - k - 2) - 1 = n - 3$. Therefore, $\Delta(\bar{G}) = n - 1 - \delta(G) \leq 2$. Combining this with Theorems 4 and 5, we have $1 \leq \Delta(\bar{G}) \leq 2$. Since $e(\bar{G}) \geq 3$, it follows that each component of \bar{G} is a path or a cycle (Note that an isolated vertex can be seen a path of order 1). From Lemma 6, the result is true for $k = 3, 4$. For $5 \leq k \leq n$, we have the following claim.

Claim 1. For any $R \subseteq V(G)$ with $|E_{\bar{G}}[w, \bar{R}]| \geq 1$ where $w \in R$ and $\bar{R} = V(G) \setminus R$, $|R| \leq 4$.

Proof of Claim 1. Assume, to the contrary, that $|R| = 5$. Set $R = \{w_i \mid 1 \leq i \leq 5\}$, and let U be the vertex set such that $|E_{\bar{G}}[w_i, U]| \geq 1$ and $|U| \leq 5$. Choose $S \subseteq V(G) \setminus R$ and $|S| = k$ such that S contains the vertex set U . Let $S = \{u_1, u_2, \dots, u_k\}$ and $\bar{S} = V(G) \setminus S = \{w_1, w_2, \dots, w_{n-k}\}$. Note that $|E_{\bar{G}}[w_i, \bar{R}]| \geq 1$ for any w_i ($1 \leq i \leq 5$). Observe that any pedant S -Steiner tree containing each vertex w_i ($1 \leq i \leq 5$) must occupy at least two vertices of \bar{S} . Another fact is that from the definition of $\tau_k(G)$, any pedant S -Steiner tree must use at least one vertex of \bar{S} . So the total number of the internally disjoint pedant S -Steiner trees is at most $2 + (n - k - 5) = n - k - 3$, a contradiction. So $|R| \leq 4$ for $5 \leq k \leq n$.

Conversely, suppose G satisfies the condition of this theorem for $5 \leq k \leq n$. It is clear that we only need to prove that $\tau_k(G) \geq n - k - 2$ where $G = K_n \setminus M$ such that $1 \leq \Delta(\bar{G}) \leq 2$ and for any $R \subseteq V(G)$ with $|E_{\bar{G}}[w, \bar{R}]| \geq 1$ where $w \in R$ and $\bar{R} = V(G) \setminus R$, the size of R is exactly 4. Set $R = \{w_i \mid 1 \leq i \leq 4\}$. From the definition of $\tau_k(G)$, it suffices to show that $\tau(S) \geq n - k - 2$ for any $S \subseteq V(G)$ with $|S| = k$. Since $|R| = 4$, we can assume $w_j u_i \in E(G)$ for any $1 \leq i \leq k$ and $5 \leq j \leq n - k$. Then the trees T_j induced by the edges in $\{u_j u_1, u_j u_2, \dots, u_j u_k\}$ ($5 \leq j \leq n - k$) form $n - k - 4$ pedant S -Steiner trees. Recall that for any $R \subseteq V(G)$ with $|E_{\bar{G}}[w, \bar{R}]| \geq 1$ where $w \in R$ and $\bar{R} = V(G) \setminus R$, $|R| \leq 4$. Therefore, there are at most four vertices in S , without loss of generality, say u_1, u_2, u_3, u_4 such that $|E_{\bar{G}}[u_i, \bar{R}]| \geq 1$ for any u_i ($1 \leq i \leq 4$). Then $w_j u_i \in E(G)$ for any $4 \leq i \leq k$ and $1 \leq j \leq 4$. Since $1 \leq \Delta(\bar{G}) \leq 2$, it follows that we may assume that $d_{\bar{G}}(w_j) = 2$ for $1 \leq j \leq 4$. Without loss of generality, let $w_1 u_1, w_1 u_2 \in M$. Since $\Delta(\bar{G}) \leq 2$, it follows that there exists a vertex in $\{w_2, w_3, w_4\}$, say w_4 , such that $w_4 u_1 \notin M$ and $w_4 u_2 \notin M$. Furthermore, $w_4 u_1 \in E(G)$, $w_4 u_2 \in E(G)$ and hence the trees T_1 induced by the edges in $\{w_4 u_1, w_4 u_2, w_4 w_1, w_1 u_3, \dots, w_1 u_k\}$ is a

pedant S -Steiner tree. Since $\Delta(\bar{G}) \leq 2$, we only need to consider the case $u_1w_2, u_2w_3 \in M$ and the case $u_1w_2, u_2w_2 \in M$. For the former case, the trees T_2 induced by the edges in $\{w_3u_1, w_2w_3, w_2u_2, \dots, w_2u_k\}$ is a pedant S -Steiner tree. For the latter case, the trees T_2 induced by the edges in $\{w_3u_1, w_3u_2, w_2w_3, w_3u_3, \dots, w_2u_k\}$ is a pedant S -Steiner tree. Therefore, the trees T_1, T_2 together with the trees T_5, \dots, T_{n-k} are $n - k - 2$ internally disjoint pedant S -Steiner trees. From the above argument, we conclude that for any $S \subseteq V(G)$ with $|S| = k$, $\tau(S) \geq n - k - 2$. From the arbitrariness of S , we have $\tau_k(G) \geq n - k - 2$. Combining this with Theorem 5, $\tau_k(G) = n - k - 2$. \blacksquare

If $k = n - 1$, then $0 \leq \tau_{n-1}(G) \leq 1$ by Proposition 1.

Corollary 2 *Let G be a connected graph of order n . Then*

- (1) $\tau_{n-1}(G) = 1$ if and only if G is a complete graph of order n .
- (2) $\tau_{n-1}(G) = 0$ if and only if G is not a complete graph of order n .

If $k = n - 2$, then $0 \leq \tau_{n-2}(G) \leq 2$ by Proposition 1.

Corollary 3 *Let G be a connected graph of order n . Then*

- (1) $\tau_{n-2}(G) = 2$ if and only if G is a complete graph of order n .
- (2) $\tau_{n-2}(G) = 1$ if and only if $G = K_n \setminus M$ and $1 \leq |M| \leq 2$, where M is a matching of K_n for $n \geq 7$.
- (3) $\tau_{n-2}(G) = 0$ if and only if G is one of the other graphs.

5 Graphs with small pedant tree-connectivity

Given a vertex x and a set U of vertices, an (x, U) -fan is a set of paths from x to U such that any two of them share only the vertex x . The size of an (x, U) -fan is the number of internally disjoint paths from x to U .

Lemma 7 (Fan Lemma, [34], p-170) *A graph is k -connected if and only if it has at least $k + 1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an (x, U) -fan of size k .*

We now turn our attention to characterize graphs with $\tau_k(G) = 0$.

Theorem 7 *Let k, n be two integers with $3 \leq k \leq n$. Let G be a connected graph of order n . Then $\tau_k(G) = 0$ if and only if G satisfies one of the following conditions.*

- (1) $0 \leq \kappa(G) \leq k - 2$;

(2) $\kappa(G) = \delta(G) = k - 1$;

(3) $\kappa(G) = k - 1$, $\delta(G) \geq k$, and there exists a vertex subset S of $V(G)$ with $|S| = k$ such that for any $S' \subseteq S$ with $|S'| = k - 1$, for any vertex $x \in V(G_i) \setminus S$, and any (x, S') -fan, u_1 is an internal vertex of some path belonging to this (x, S') -fan, where G_i is the connected component of $G \setminus S'$ containing u_1 .

Proof. Suppose $\tau_k(G) = 0$. If $k = n$, then $\tau_n(G) = 0$ if and only if G is a connected graph by Corollary 1, which implies that $\tau_n(G) = 0$ if and only if $0 \leq \kappa(G) \leq n - 2$ or $\kappa(G) = \delta(G) = n - 1$, as desired. We now assume $3 \leq k \leq n - 1$. Then we have the following claim.

Claim 1. $\kappa(G) \leq k - 1$.

Proof of Claim 1. Assume, to the contrary, that $\kappa(G) \geq k$. For any $S \subseteq V(G)$ and $|S| = k$, there is a vertex $x \in V(G) \setminus S$ since $3 \leq k \leq n - 1$. Let $S = \{u_1, u_2, \dots, u_k\}$. Since $\kappa(G) \geq k$, it follows from Lemma 7 that there exists an (x, S) -fan of size k in G . Let P_1, P_2, \dots, P_k be the k internally disjoint paths of this (x, S) -fan. Then the tree T induced by the edges in $E(P_1) \cup E(P_2) \cup \dots \cup E(P_k)$ is a pedant S -Steiner tree, which implies that $\tau(S) \geq 1$. From the arbitrariness of S , we have $\tau_k(G) \geq 1$, a contradiction. ■

From Claim 1, we have $0 \leq \kappa(G) \leq k - 1$. If $0 \leq \kappa(G) \leq k - 2$, then (1) holds. If $\kappa(G) = k - 1$, then $\delta(G) \geq \kappa(G) = k - 1$. Furthermore, if $\kappa(G) = \delta(G) = k - 1$, then (2) holds. From now on, we assume that $\kappa(G) = k - 1$ and $\delta(G) \geq k$.

Claim 2. There exists a vertex subset S of $V(G)$ with $|S| = k$ such that for any $S' \subseteq S$ with $|S'| = k - 1$, if S' is not a vertex cut set of G , then for any vertex $x \in V(G) \setminus S$, and any (x, S') -fan, u_1 is an internal vertex of a path belonging to this (x, S') -fan.

Proof of Claim 2. Assume, to the contrary, that for any $S \subseteq V(G)$ with $|S| = k$, there exists a vertex subset S' in S with $|S'| = k - 1$ such that S' is not a vertex cut of G , and there exists a vertex $x \in V(G) \setminus S$ and an (x, S') -fan, u_1 is not an internal vertex of for any path of this (x, S') -fan. Let $S' = \{u_2, u_3, \dots, u_k\}$ and $S \setminus S' = \{u_1\}$, where $S = \{u_1, u_2, \dots, u_k\}$. Since $\kappa(G) = k - 1$, it follows from Lemma 7 that there is an (x, S') -fan in G , where $x \in V(G) \setminus S$. Note that $x \neq u_1$. Denote by P_2, P_3, \dots, P_{k-1} the k internally disjoint paths connecting x and u_2, u_3, \dots, u_{k-1} of this (x, S') -fan, respectively. Recall that u_1 is not an internal vertex of for any path of this (x, S') -fan. Since $\kappa(G) = k - 1$ and S' is not a vertex cut of G , it follows that $G \setminus S'$ is connected and hence there is a path connecting x and u_1 , say P_1 . Clearly, the graph H induced by the edges in $E(P_1) \cup E(P_2) \cup \dots \cup E(P_k)$ contains a pedant S -Steiner tree, which implies that $\tau(S) \geq 1$. From the arbitrariness of S , we have $\tau_k(G) \geq 1$, a contradiction. ■

Furthermore, we have the following claim.

Claim 3. There exists a vertex subset S of $V(G)$ with $|S| = k$ such that for any $S' \subseteq S$ with $|S'| = k - 1$, if S' is a vertex cut set of G , then for any vertex $x \in V(G_i) \setminus u_1$, and any (x, S') -fan, u_1 is an internal vertex of a path belonging to this (x, S') -fan, where G_i is the connected component of $G \setminus S'$ containing u_1 .

Proof of Claim 3. Assume, to the contrary, that for any $S \subseteq V(G)$ with $|S| = k$, there exists a vertex subset S' in S with $|S'| = k - 1$ such that S' is a vertex cut of G , and there exist a vertex $x \in V(G_i) \setminus u_1$ and an (x, S') -fan such that u_1 does not belong to this (x, S') -fan, where G_i is the connected component of $G \setminus S'$ containing u_1 . Denote by P_2, P_3, \dots, P_{k-1} the k internally disjoint paths connecting x and u_2, u_3, \dots, u_{k-1} of this (x, S') -fan, respectively. Since G_i is connected, there is a path connecting x and u_1 , say P_1 . Clearly, the graph H induced by the edges in $E(P_1) \cup E(P_2) \cup \dots \cup E(P_k)$ contains a pedant S -Steiner tree, which implies that $\tau(S) \geq 1$. From the arbitrariness of S , we have $\tau_k(G) \geq 1$, a contradiction. ■

From Claims 2 and 3, we know that (3) holds.

Conversely, we suppose that G is a graph satisfying one of conditions in this theorem. Our aim is to show $\tau_k(G) = 0$. Suppose $0 \leq \kappa(G) \leq k - 2$. If $\tau_k(G) \geq 1$, then we have $\kappa(G) \geq k - 1$ by Lemma 1, a contradiction. Therefore, $\tau_k(G) = 0$, as desired. Suppose $\kappa(G) = \delta(G) = k - 1$. Then there exists a vertex of degree $k - 1$, say u_1 . Let $N_G(u_1) = \{u_2, u_3, \dots, u_k\}$. Choose $S = \{u_1\} \cup N_G(u_1)$. Clearly, there is no pedant S -Steiner tree in G . Hence $\tau_k(G) = 0$, as desired. Suppose that G is a graph satisfying Condition (3). For the vertex set $S \subseteq V(G)$, there is no pedant S -Steiner tree in G , and hence $\tau_k(G) = 0$. ■

6 Nordhaus-Guddum type result

In this section, we study the Nordhaus-Gaddum type relations for pedant-tree connectivity.

Theorem 8 *Let k, n be two integers with $3 \leq k \leq n$, and let G be a connected graph of order n . Then*

$$(1) \quad 0 \leq \tau_k(G) + \tau_k(\bar{G}) \leq n - k;$$

$$(2) \quad 0 \leq \tau_k(G) \cdot \tau_k(\bar{G}) \leq \left\lfloor \frac{n-k}{2} \right\rfloor^2.$$

Moreover, the upper and lower bounds are sharp.

Proof. (1) To avoid confusion, we denote the local pedant tree-connectivity of a k -subset S in a graph G by $\tau(G; S)$. Since $G \cup \bar{G} = K_n$, for any k -subset S we have $\tau(G; S) + \tau(\bar{G}; S) \leq$

$\tau(K_n; S)$. Suppose that $\tau_k(K_n) = \tau(K_n; S_0)$ for some k -subset S_0 . Then we have

$$\tau_k(K_n) = \tau(K_n; S_0) \geq \tau(G; S_0) + \tau(\bar{G}; S_0) \geq \tau_k(G) + \tau_k(\bar{G}).$$

This together with $\tau_k(K_n) = n - k$ results in $\tau_k(G) + \tau_k(\bar{G}) \leq n - k$.

(2) It follows immediately from (1).

Example 1: Let G' be a graph of order $n - 4$, and let $v_1v_2v_3v_4$ be a path. Let G be the graph obtained from G' and the path by adding edges between the vertex v_1 and all vertices of G' and adding edges between the vertex v_4 and all vertices of G' . Since $\delta(G) = \delta(\bar{G}) = 2$, it follows that $\tau_k(G) = \tau_k(\bar{G}) = 0$. So the lower bound of Theorem 8 is sharp for $3 \leq k \leq n$. From Proposition 2, if $G = K_n$, then $\tau_k(G) = n - k$ and $\tau_k(\bar{G}) = 0$, and hence $\tau_k(G) + \tau_k(\bar{G}) = n - k$. So the upper bound of Theorem 8 is sharp for $3 \leq k \leq n$.

Let us focus on (1) of Theorem 8. If one of G and \bar{G} is disconnected, we can characterize the graphs attaining the upper bound by Lemma 4.

Proposition 2 *For any graph G of order n , if G is disconnected, then $\tau_k(G) + \tau_k(\bar{G}) = n - k$ if and only if $\bar{G} = K_n$.*

If both G and \bar{G} are all connected, we can obtain a structural property of the graphs attaining the upper bound.

Proposition 3 *If $\tau_k(G) + \tau_k(\bar{G}) = n - k$, then $\Delta(G) - \delta(G) \leq k - 1$.*

Proof. Assume that $\Delta(G) - \delta(G) \geq k$. Since $\tau_k(\bar{G}) \leq \delta(\bar{G}) = n - 1 - \Delta(G)$, $\tau_k(G) + \tau_k(\bar{G}) \leq \delta(G) + n - 1 - \Delta(G) \leq n - 1 - k$, a contradiction. ■

One can see that the graphs with $\tau_k(G) + \tau_k(\bar{G}) = n - k$ must have a uniform degree distribution.

From Corollary 1, the following observation are immediate.

Observation 3 *Let G be a graph of order n ($n \geq 3$). Then $\tau_n(G) + \tau_n(\bar{G}) = 0$ if and only if G is a graph of order n .*

From Corollary 2 and Theorem 8, we have the following result.

Observation 4 *Let G be a graph of order n ($n \geq 4$). Then*

$$0 \leq \tau_{n-1}(G) + \tau_{n-1}(\bar{G}) \leq 1.$$

Furthermore, $\tau_{n-1}(G) + \tau_{n-1}(\bar{G}) = 1$ if and only if G or \bar{G} is complete; $\tau_{n-1}(G) + \tau_{n-1}(\bar{G}) = 0$ if and only if both G and \bar{G} are not complete.

From Corollary 3 and Theorem 8, we have the following result.

Proposition 4 *Let G be a graph of order n ($n \geq 5$). Then*

$$\tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 0 \text{ or } \tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 2.$$

Furthermore, $\tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 2$ if and only if G or \bar{G} is complete; $\tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 0$ if and only if both G and \bar{G} are not complete.

Proof. From Theorem 8,

$$0 \leq \tau_{n-2}(G) + \tau_{n-2}(\bar{G}) \leq 2.$$

Suppose $\tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 1$. Then $\tau_{n-2}(G) = 1$ or $\tau_{n-2}(\bar{G}) = 1$. Without loss of generality, let $\tau_{n-2}(G) = 1$ and $\tau_{n-2}(\bar{G}) = 0$. From Corollary 3, G or \bar{G} is a graph obtained from a complete graph of order n by deleting at most two edges. Therefore, $e(G) = \binom{n}{2} - x$ and $e(\bar{G}) = \binom{n}{2} - y$, where $1 \leq x, y \leq 2$. Since $e(G) + e(\bar{G}) = \binom{n}{2}$, it follows that $2 \leq x + y = \binom{n}{2} \leq 4$ and hence $n = 2$ or $n = 3$, a contradiction. Hence

$$\tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 0 \text{ or } \tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 2.$$

Suppose $\tau_{n-2}(G) + \tau_{n-2}(\bar{G}) = 2$. Then $\tau_{n-2}(G) = 2$ or $\tau_{n-2}(\bar{G}) = 2$. Without loss of generality, let $\tau_{n-2}(G) = 2$ or $\tau_{n-2}(\bar{G}) = 0$. From Corollary 3, graph G is complete. ■

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